Reflected Backward Stochastic Differential Equations with Continuous Coefficient and L^2-Barriers

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Abstract

In this paper we study reflected backward stochastic differential equations with a continuous, linear growth coefficient and two barriers which belong to L^2. We prove that there exists at least by penalization method.

Keywords: Backward stochastic differential equation; reflected barrier; penalization method

1 Introduction

Since Pardoux and Peng [8] introduced nonlinear backward stochastic differential equations (BSDEs for short) with Lipschitz coefficient, there follows many results in this topic. Lepeltier and San Martin [3] studied BSDEs with continuous coefficient, they proved that in this case there exists at least one but not necessarily unique solution. Lin and Peng. [6] got g-supersolution for BSDEs with continuous drift coefficient. El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [1] considered reflected backward stochastic differential equations (RBSDEs for short) for the first time, that is to say the solution should be above or below some given process. They proved that if the coefficient is Lipschitz and the lower barrier is continuous , then there exists a unique solution. And then, Lepeltier and San Martin [4] studied BSDEs with continuous coefficient and two continuous barriers. In Hamadène [7], he studied the case of a right-continuous with left limits barrier (R.C.L.L. for short). Recently, Lepeltier and Xu [5] gave the results of BSDEs with Lipschitz coefficient and R.C.L.L. barriers, and then in Peng and Xu [10] with L^2-barriers.

In this paper, we work on BSDEs with continuous coefficient and two L^2-barriers. We apply the result in Lepeltier and San Martin [3], which showed that for a continuous function f, there exists a sequence of Lipschitz function f_m that converges to f as $m \to \infty$, to deal with the continuous coefficient. The penalization method is employed to tackle the L^2-barriers. Our proof is also based on the monotonic limit theorem in Peng [9].

This paper is organized as follows: in section 2, we formulate the problem for the solutions of RBSDEs with two L^2-barriers. In section 3, some prelilinary results are given which will be used in the proof. Then in the last section, we give the proof of existence of solution for RBSDEs with two L^2-barriers.

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2 Formulation of the Problem

On a given complete probability space $(\Omega,\mathscr{F},\mathrm{P})$, $\{B_t, 0 \le t \le T\}$ is the d-dimensional standard Brownain motion, $\{\mathscr{F}_t, 0 \le t \le T\}$ is the augmentation of the natural filtration generated by the Brownain motion.

We introduce the following spaces:

- $L^2 = \{\xi : \Omega \to \mathbb{R}^d,\ \mathscr{F}_T\text{-measuable random variable with } \mathrm{E}[|\xi|^2] < \infty\}$;
- $L^2_{\mathscr{F}} = \{\varphi : \Omega \times [0,t] \to \mathbb{R}^d,\ \mathscr{F}_t\text{-measuable process with } \mathrm{E}[\int_0^t |\varphi_t|^2 \mathrm{d}t] < \infty\}$;
- $S^2_{\mathscr{F}} = \{\varphi \in L^2_{\mathscr{F}} : \text{progressively measurable R.C.L.L. process with } \mathrm{E}[\sup_{0\le t\le T} |\varphi_t|^2] < \infty\}$.

First of all we give the following assumptions:

Assumption 1. The terminal value ξ is in L^2.

Assumption 2. The function $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to f(t,w,y,z)$, for any $(t,w) \in [0,T] \times \Omega$, $f(t,w,y,z)$ is continuous on $\mathbb{R} \times \mathbb{R}^d$, P-almost surely. And there exists a constant K, such that for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,w,y,z)| \le K(1 + |y| + |z|) \quad \text{P-a.s.}$$

Assumption 3. The barriers $L, U \in L^2_{\mathscr{F}}$ satisfy:

$$\mathrm{E}[\operatorname{ess}\sup_{0\le t\le T} (L_t^+)^2] < \infty,\ \mathrm{E}[\operatorname{ess}\sup_{0\le t\le T} (U_t^+)^2] < \infty,$$

$$L_T \le \xi \le U_T \ \text{ a.s.}, \qquad L_t \le U_t \ \text{ for all } t \in [0,T].$$

Assumption 4. There exists a process

$$X_t^0 = X_0^0 + A_t^0 - K_t^0 + \int_0^t Z_s^0 \mathrm{d}B_s \qquad 0 \le t \le T\,, \tag{1}$$

with $Z^0 \in L^2_{\mathscr{F}}$, $A^0, K^0 \in S^2_{\mathscr{F}}$, and increasing with $A_0^0 = K_0^0 = 0$, such that $L_t \le X_t^0 \le U_t$ a.e. a.s.

We introduce the definition of the solution for RBSDE with two barriers L, U:

Definition 2.1 *A quadruple $(Y,Z,A,K) \in S^2_{\mathscr{F}} \times L^2_{\mathscr{F}} \times S^2_{\mathscr{F}} \times S^2_{\mathscr{F}}$ is called a solution for RBSDE with the lower barrier $L \in L^2_{\mathscr{F}}$, the upper barrier $U \in L^2_{\mathscr{F}}$, the terminal condition $\xi \in L^2$ and the coefficient f if it satisfies:*

1. *A, K are increasing.*
2. *(Y,Z,A,K) satisfies the following BSDE*

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + A_T - A_t - K_T + K_t - \int_t^T Z_s dB_s, \quad 0 \le t \le T. \tag{2}$$

3. *$L_t \le Y_t \le U_t$, a.e. a.s.*

4. *Generalized Skorohod condition:*
   for each $L^*, U^* \in S^2_{\mathscr{F}}$ *such that* $L_t \le L^*_t \le Y_t \le U^*_t \le U_t$ *a.e. a.s.,*

$$\int_0^T (Y_{s-} - L^*_{s-})dA_s = \int_0^T (U^*_{s-} - Y_{s-})dK_s = 0. \tag{3}$$

In this paper, our main result is the following Theorem 2.2 which will be proved in section 4.

Theorem 2.2 *Under Assumptions (1),(2),(3),(4), there exists at least one solution* (Y, Z, A, K) *for RBSDEs with two* L^2*-barriers.*

3 Some Preliminary Results

In this section, we introduce some preliminary definitions and results that will be used later. We first introduce g-supersolution which is very important for the prove of the existence theorem:

Definition 3.1 *(See Peng [9], El Karoui et al. [2])We call a triple* $(Y, Z, A) \in S^2_{\mathscr{F}} \times L^2_{\mathscr{F}} \times S^2_{\mathscr{F}}$ *a g-supersolution if A is an increasing process in* $S^2_{\mathscr{F}}$ *and the triple satisfies:*

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dB_s \qquad t \in [0, T]. \tag{4}$$

For a continuous function with linear growth, we have the following lemma:

Lemma 3.2 *(See Lepeltier and San Martin [3]) let* $f : \mathbb{R}^p \to \mathbb{R}, p \in \mathbb{N}$, *be a continuous function with linear growth, that is to say* $\forall x \in \mathbb{R}^p$, $|f(x)| \le K(1 + |x|)$. *Define* $f_m(x) = \inf_{y\in\mathbb{Q}^p}(f(y) + m|x - y|)$, *then for* $m > K$, $f_m : \mathbb{R}^p \to \mathbb{R}$ *satisfies:*

1. *Linear growth:* $\forall x \in \mathbb{R}^p$, $|f_m(x)| \le K(1 + |x|)$;
2. *Monotonicity:* $\forall x \in \mathbb{R}^p$, $f_m(x) \uparrow f(x)$;
3. *Lipschitz condition:* $\forall x \in \mathbb{R}^p$, $|f_m(x) - f_m(y)| \le m|x - y|$;
4. *Strong convergence: if* $x_m \to x$, *then* $f_m(x_m) \to f(x)$.

The following generalized Monotonic Limit Theorem of BSDEs is proved in Peng and Xu [10].

Consider the following sequence of Itô's process:

$$y^i_t = y^i_0 + \int_0^t g^i_s s - A^i_t + K^i_t + \int_0^t z^i_s B_s, \quad i = 1, 2, \cdots . \tag{5}$$

here for each $i,$ the process $g^i \in L^2_{\mathscr{F}},\ A^i, K^i \in S^2_{\mathscr{F}}$ are given, and $\{A^i, K^i\}_{i=1}^{\infty}$ satisfies

(i) A^i is continuous and increasing such that $A^i_0 = 0$ and $\mathrm{E}(A^i_T)^2 < \infty$.

(ii) K^i is increasing and $K^i_0 = 0$.

(iii) $K^i_t - K^i_s \geq K^j_t - K^j_s \quad \forall 0 \leq s \leq t \leq T \quad a.s., \quad \forall i \geq j.$

(iv) for each $t \in [0,T]$, $K^j_t \uparrow K_t$ with $\mathrm{E}[K^2_T] < \infty$.

Furthermore, we assume that

(v) $\{g^i, z^i\}_{i=1}^{\infty}$ converges weakly to (g^0, z) in $L^2_{\mathscr{F}}$.

(vi) $\{y^i_t\}_{i=1}^{\infty}$ converges increasingly to (y_t) with $\mathrm{E}[\sup\limits_{0\leq t\leq T} |y_t|^2] < \infty$.

Theorem 3.3 *Let the above assumptions hold, we have the limit of* $\{y^i_t\}_{i=1}^{\infty}$ (y_t) *has a form* $y_t = y_0 + \int_0^t g^0_s s - A_t + K_t + \int_0^t z_s B_s$, *where* A *and* K *are increasing processes in* $S^2_{\mathscr{F}}$. *For each* $t \in [0,T]$, A_t*(resp.*K_t*) is the weak(resp.strong) limit of* $\{A^i_t\}_{i=1}^{\infty}$*(resp.*$\{K^i_t\}_{i=1}^{\infty}$*). Furthermore for any* $p \in [1,2)$, $\{z^i_t\}_{i=1}^{\infty}$ *converges strongly to* z_t *in* $L^p_{\mathscr{F}}$.

4 Proof of the Main Result

In this section we prove Theorem 2.2, i.e. the existence for the solution of RBSDEs with two L^2-barriers. Firstly, we consider, for any integer $m,$ the following RBSDEs with a upper barrier U:

$$Y^m_t = \xi + \int_t^T f_m(s, Y^m_s, Z^m_s)\mathrm{d}s + m\int_t^T (L_s - Y^m_s)^+\mathrm{d}s - K^m_T + K^m_t - \int_t^T Z^m_s \mathrm{d}B_s. \quad (6)$$

Since the coefficient are Lipschitz, according to Peng and Xu [10] these equations have unique solutions $(Y^m, Z^m, K^m), \forall m \in \mathbb{N}$.

Then for any $n, m \geq 1$, we consider the following classical BSDEs:

$$Y^{n,m}_t = \xi + \int_t^T f_m(s, Y^{n,m}_s, Z^{n,m}_s)\mathrm{d}s - \int_t^T Z^{n,m}_s \mathrm{d}B_s + m\int_t^T (L_s - Y^{n,m}_s)^+ s - n\int_t^T (Y^{n,m}_s - U_s)^+\mathrm{d}s. \quad (7)$$

since $g_{n,m}(t,y,z) = f_m(t,y,z) + m(L_t - y)^+ - n(y - U_t)^+$ are Lipschitz in (y,z), uniformly in (t,w), the equations have unique solutions $(Y^{n,m}, Z^{n,m})$. And by comparison theorem , we have that for fixed n, $Y^{n,m}$ is increasing in m.

Set $A^{n,m}_t = m\int_0^t (L_s - Y^{n,m}_s)^+\mathrm{d}s, \ K^{n,m}_t = n\int_0^t (Y^{n,m}_s - U_s)^+\mathrm{d}s,$ we have the following proposition:

Proposition 4.1 *There exists a constant* C *independent on* n, m *such that*

$$E[\sup_{0\leq t\leq T} (Y^{n,m}_t)^2] + E[\int_0^T |Z^{n,m}_t|^2 ds] + E[(A^{n,m}_T)^2] + E[(K^{n,m}_T)^2] \leq C. \quad (8)$$

To prove this result, we need the following two lemmas.

Consider the following equation:

$$Y_t^m = \xi + \int_t^T f_m(s, Y_s^m, Z_s^m)\mathrm{d}s + m\int_t^T (L_s - Y_s^m)^+\mathrm{d}s - \int_t^T Z_s^m \mathrm{d}B_s. \tag{9}$$

this is a sequence of classical BSDE, there exists unique solutions (Y^m, Z^m), for all $m \in \mathbb{N}$.

Lemma 4.2 *For equation (9), we have that there exists a constant C independent of m such that*

$$E[\sup_{0\le t\le T}(Y_t^m)^2] + E[\int_0^T |Z_s^m|^2 ds] + E[(A_T^m)^2] \le C. \tag{10}$$

where $A_t^m := m\int_0^t (L_s - Y_s^m)^+ ds$.

Apply the Itô's formula on $(Y_t^m)^2$, the conclusion can be deduced owe to the Gronwall's lemma and B-D-G inequality.

We can easily get a similarly result as Lemma 5.1 in Peng and Xu [10]:

Lemma 4.3 *There exists a quadruple $(Y^*, Z^*, A^*, K^*) \in S^2_{\mathscr{F}} \times L^2_{\mathscr{F}} \times S^2_{\mathscr{F}} \times S^2_{\mathscr{F}}$ such that*

$$Y_t^* = \xi + \int_t^T f(s, Y_s^*, Z_s^*)ds + A_T^* - A_t^* - (K_T^* - K_t^*) - \int_t^T Z_s^* dB_s. \tag{11}$$

where A^*, K^* are both increasing, and $L_t \le Y_t^* \le U_t$ a.e.,a.s.

Proof of Proposition 4.1: Let (Y^+, Z^+) and (Y^-, Z^-) be the solution of the following two BSDEs:

$$Y_t^+ = \xi + \int_t^T f_m(s, Y_s^+, Z_s^+)s + A_T^* - A_t^* + m\int_t^T (L_s - Y_s^+)^+ s - \int_t^T Z_s^+ B_s. \tag{12}$$

$$Y_t^- = \xi + \int_t^T f_m(s, Y_s^-, Z_s^-)s - (K_T^* - K_t^*) - n\int_t^T (Y_s^- - U_s)^+ s - \int_t^T Z_s^- B_s. \tag{13}$$

where (A^*, K^*) is given as in Lemma 4.3. From the comparison theorem of the standard BSDE, we have:

$$Y_t^- \le Y_t^{n,m} \le Y_t^+ \quad \forall t \in [0,T] \ \text{ a.s.}$$

Review Lemma 4.2, obviously we can prove the same result if we replace f_m by f, or replace $m\int_t^T (L_s - Y_s^m)^+\mathrm{d}s$ by $-n\int_t^T (Y_s^m - U_s)^+\mathrm{d}s$, so we have:

$$\mathrm{E}[\sup_{0\le t\le T}(Y_t^+)^2] + \mathrm{E}[\sup_{0\le t\le T}(Y_t^-)^2] \le C,$$

then

$$\mathrm{E}[\sup_{0\le t\le T}(Y_t^{n,m})^2] \le \max\{\mathrm{E}[\sup_{0\le t\le T}(Y_t^+)^2],\, \mathrm{E}[\sup_{0\le t\le T}(Y_t^-)^2]\} \le C.$$

For $A_T^{n,m}$, we consider the following BSDE:

$$\tilde{Y}_t^m = \xi + \int_t^T f_m(s, \tilde{Y}_s^m, \tilde{Z}_s^m)s - (K_T^* - K_t^*) + m\int_t^T (L_s - \tilde{Y}_s^m)^+ s - \int_t^T \tilde{Z}_s^m B_s. \tag{14}$$

We know Y^* satisfies $L_t \leq Y_t^* \leq U_t$ from Lemma 4.3, thus we can add the zero term $m\int_t^T(L_s - Y_s^*)^+ s$ to the right side of (11). Since $A_t^* \geq 0$, from the comparison theorem, it follows that $Y_t^* \geq \tilde{Y}_t^m$, thus $U_t \geq \tilde{Y}_t^m$, then $-m\int_t^T(\tilde{Y}_s^* - U_s)^+ s$ is zero and so can be added to the right side of (14). Again from the comparison theorem, we derive $\tilde{Y}_t^m \leq Y_t^{n,m}$, and so:

$$0 \leq A_t^{n,m} \leq \tilde{A}_t^m := m\int_0^t (L_s - \tilde{Y}_s^m)^+ s,$$

then following the same process as Lemma 4.2, we have $\mathrm{E}(A_T^{n,m})^2 \leq \mathrm{E}(\tilde{A}_T^m)^2 \leq C$.

Now we consider the BSDE

$$\tilde{Y}_t^n = \xi + \int_t^T f(s, \tilde{Y}_s^n, \tilde{Z}_s^n) s + A_T^* - A_t^* - n\int_t^T (\tilde{Y}_s^n - U_s)^+ s - \int_t^T \tilde{Z}_s^n B_s. \tag{15}$$

Similarly, we can get $E(K_T^{n,m})^2 \leq C$.

Apply Itô's formula to $(Y_t^{n,m})^2$, we have:

$$\begin{aligned}
&\mathrm{E}|Y_t^{n,m}|^2 + \mathrm{E}\int_t^T |Z_s^{n,m}|^2 s \\
&\leq C(1 + \mathrm{E}\int_t^T |Y_s^{n,m}|^2 \mathrm{d}s) + \alpha \mathrm{E}\int_t^T |Z_s^{n,m}| s + \beta \mathrm{E}[ess \sup_{0\leq t\leq T} (L_t^+)^2] \\
&+ \gamma \mathrm{E}[ess \sup_{0\leq t\leq T} (U_t^-)^2] + \frac{1}{\beta}\mathrm{E}(A_T^{n,m})^2 + \frac{1}{\gamma}\mathrm{E}(K_T^{n,m})^2.
\end{aligned}$$

choose $\alpha = \frac{1}{3}$, we get $\mathrm{E}\int_0^T |Z_s^{n,m}|^2 s \leq C$. The proof of Proposition 4.1 is completed.

□

To prove the Theorem 2.2, we let n tend to ∞, then

$$\begin{cases}
Y^{n,m} \to Y^m & \text{in } L^2_{\mathscr{F}}. \\
n\int_0^T (Y_s^{n,m} - U_s)^+ s \to K_T^m & \text{in } L^2. \\
Z^{n,m} \to Z^m & \text{in } S^2_{\mathscr{F}}.
\end{cases}$$

where (Y^m, Z^m, K^m) is the unique solution of the following RBSDE:

$$Y_t^m = \xi + \int_t^T f_m(s, Y_s^m, Z_s^m) s - (K_T^m - K_t^m) - m\int_t^T (L_s - Y_s^m)^+ s - \int_t^T Z_s^m B_s. \tag{16}$$

we know that $Y^m \leq U$ a.e. a.s..

And we also have the following lemma:

Lemma 4.4 *There exists a constant C independent on m, such that*

$$E[\sup_{0\leq t\leq T} (Y_t^m)^2] + E[\int_0^T |Z_t^m|^2 ds] + E[(A_T^m)^2] + E[(K_T^m)^2] \leq C. \tag{17}$$

where $A_t^m = m\int_0^t (L_s - Y_s^m)^+ ds$.

From the comparison theorem, Y^m is increasing in m, so there exists a process Y such that $Y^m \uparrow Y$, and from Fatou's lemma $\mathrm{E}[\sup_{0\le t\le T} Y_t^2] \le C$.

By the dominated convergence theorem it follows that

$$\mathrm{E}\int_0^T (Y_t - Y_t^m)^2 \mathrm{d}t \to 0, \qquad \text{as } n \to \infty.$$

We have already get the conclusion that (Y^m, Z^m) is the solution of (16). Rewrite (16) in a forward version:

$$Y_t^m = Y_0^m + \int_0^t f_m(s, Y_s^m, Z_s^m)s - A_t^m + K_t^m - \int_0^t Z_s^m B_s. \tag{18}$$

Set $g_t^m = -f_m(s, Y_s^m, Z_t^m)$, with Lemma 4.4, we derive that all assumptions of Theorem 3.3 are satisfied. It follows that its limit Y is in $S_{\mathscr{F}}^2$ and has the form

$$Y_t = \xi + \int_t^T g_s^0 s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s B_s. \tag{19}$$

where (g^0, Z, A) is the weak limit of $\{g(\cdot, Y^m, Z^m), Z^m, A^m\}_{i=1}^\infty$ in $L_{\mathscr{F}}^2$, K is the strong limit of $\{K_t^m\}_{i=1}^\infty$ in $L_{\mathscr{F}}^2$, A and K are increasing processes in $S_{\mathscr{F}}^2$. Furthermore, for any $p \in [1,2)$, we have $\lim\limits_{m\to\infty} E\int_0^T |Z_s^m - Z_s|^p s = 0$. In Lemma 3.2, we showed that the sequence of Lipschitz function f_m converges strongly to the continuous function f, so we get $f_m(\cdot, Y^m, Z^m) \to f(\cdot, Y, Z)$ because of the strong convergence of Y^m to Y and the weak convergence of Z^m to Z, and then:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s B_s. \tag{20}$$

The only problem left is to verify the generalized Skorohod condition . For the upper barrier U, it is easily to prove that for any $U^* \in S_{\mathscr{F}}^2$ and $Y \le U^* \le U$, if we consider large enough m, then $Y^m \le U^* \le U$. For the solution of the RBSDE (16), we get $\int_0^T (U_{t-}^* - Y_{t-}^m) K_t^m = 0$ from the generalized Skorokhod condition. So we get $\int_0^T (U_{t-}^* - Y_{t-}) K_t^m = 0$, since $0 \le U_{t-}^* - Y_{t-} \le U_{t-}^* - Y_{t-}^m$. Furthermore $K_T^m \uparrow K_T$, so $0 \le \int_0^T (U_{t-}^* - Y_{t-})(K_t - K_t^m) \le (K_T - K_T^m) \max_{t\in[0,T)} (U_{t-}^* - Y_{t-}) \to 0$. The Skorohod condition for the upper barrier U is obtained.

At last we prove the Skorohod condition holds for the lower barrier L. Consider the following BSDE:

$$\tilde{Y}_t^m = \xi + \int_t^T f_m(s, \tilde{Y}_s^m, \tilde{Z}_s^m)s + m\int_t^T (L_s - \tilde{Y}_s^m)^+ s - (K_T - K_t) - \int_t^T \tilde{Z}_s^m B_s. \tag{21}$$

We denote $\bar{Y}^m := \tilde{Y}^m - K$ and rewrite the BSDE:

$$\bar{Y}_t^m = \xi - K_T + \int_t^T f_m^K(s, \bar{Y}_s^m, \tilde{Z}_s^m)s + m\int_t^T (L_s - K_s - \bar{Y}_s^m)^+ s - \int_t^T \tilde{Z}_s^m B_s. \tag{22}$$

where $f_m^K(t, y, z) := f_m(t, y + K, z)$.

If we consider a BSDE with coefficient f^K and lower barrier L^K, where $f^K(t, y, z) = f(t, y + K, z)$, $L^K = L - K$, then the BSDE above is the penalized equation of this problem, we know that it has the unique solution $(\bar{Y}^m, \tilde{Z}^m, \tilde{A}^m)$. When $m \to \infty$, we get the limit:

$$\bar{Y}_t = \xi - K_T + \int_t^T f^K(s, \bar{Y}_s, \tilde{Z}_s)s + \tilde{A}_T - \tilde{A}_t - \int_t^T \tilde{Z}_s B_s. \tag{23}$$

here $\tilde{A}_t$ is the $L^2_{\mathscr{F}}$ weak limit of $\tilde{A}^m_t = m\int_0^t (L_s - \tilde{Y}^m_s)^+ s = m\int_0^t (L_s - K_s - \bar{Y}^m_s)^+ s$. Suppose $\ddot{Y}$ is another f^K-supersolution with decomposition $(\ddot{Z}, \ddot{A})$, which satisfies (23) and $\ddot{Y}_t \geq L_t - K_t$. By comparison theorem, we have $\bar{Y}^m_t \leq \ddot{Y}_t$, so $\bar{Y}_t \leq \ddot{Y}_t$. That is to say $\bar{Y}$ is the smallest f^K-supersolution with $\bar{Y}_T = \xi - K_T$ that dominates $L - K$, and from the comparison theorem we have $Y^m_t \geq \tilde{Y}^m_t$, so we get:

$$\tilde{A}^m_t - \tilde{A}^m_s = m\int_s^t (L_r - \tilde{Y}^m_r)^+ dr = A^m_t - A^m_s \quad 0 \leq s \leq t \leq T.$$

thus $\tilde{A}_t - \tilde{A}_t \geq A_t - A_s$. From (20) we know $Y - K$ is a f^K-supersolution, compare this with (23), we have $Y - K \leq \bar{Y}$. thus $Y - K = \bar{Y}$ is the smallest f^K-supersolution with terminal condition $\xi - K_T$that dominates $L - K$. With the help of the following Proposition 4.6, we can get that for each $L^* \in S^2_{\mathscr{F}}$ such that $Y \geq L^* \geq L$, we have $Y - K \geq L^* - K \geq L - K$, then:

$$\int_0^T (Y_{t-} - L^*_{t-})A_t = \int_0^T ((Y_{t-} - K_{t-}) - (L^*_{t-} - K_{t-}))A_t = 0.$$

The proof of the existence for solution of RBSDEs is completed.

□

At last, we prove that if Y is the smallest f-supersolution that dominates L, then Y satisfies the Skorohod condition, which was used in above proof.

According to Peng and Xu [10], we have the following proposition:

Proposition 4.5 *Given $Y \in S^2_{\mathscr{F}}$, $Y_T = \xi \in L^2$, $L \in L^2_{\mathscr{F}}$, the following two items are equivalent:*

i) Y is the smallest g-supersolution that dominates L.

ii) For any $L^* \in S^2_{\mathscr{F}}$, $Y_t \geq L^*_t \geq L_t$, a.e.,a.s., Y is the smallest g-supersolution that dominates L^*.

Now we consider the following condition: $L \in S^2_{\mathscr{F}}$ is a given process, $f_0(t) \equiv 0$, $\hat{Y} \in S^2_{\mathscr{F}}$ is a f_0-supersolution that dominates L with terminal condition ξ, i.e.

$$\hat{Y}_t = \xi + A_T - A_t - \int_t^T Z_s \mathrm{d}B_s, \quad \hat{Y}_t \geq L_t, \quad \forall t \in [0,T] \ \text{ a.s.} \tag{24}$$

where (Z, A) is the corresponding composition of $\hat{Y}$. From Peng and Xu [10], we know that if $\hat{Y}$ is the smallest f_0-supersolution that dominates L with terminal condition ξ, then for each stopping time $\tau \leq T$, we have $\hat{Y}_{\tau-} = \hat{Y}_\tau \vee L_{\tau-}$. Then we have:

$$\sum_{0 \leq t \leq T} (\hat{Y}_{t-} - L_{t-})(A_t - A_{t-}) = 0 \quad \text{a.s.} \tag{25}$$

Proposition 4.6 *We claim that the following two items are equivalent:*

i) Y is the smallest f-supersolution that dominates L with terminal condition ξ.

ii) $\hat{Y}$ is the smallest f_0-supersolution that dominates $\hat{L}$ with terminal condition $\hat{\xi}$, where for each $t \in [0,T]$:

$$\hat{f}(t) := f(t, Y_t, Z_t), \quad \hat{Y}_t := Y_t + \int_0^t \hat{f}(s)ds, \quad \hat{L}_t := L_t + \int_0^t \hat{f}(s)ds, \quad \hat{\xi} := \xi + \int_0^T \hat{f}(s)ds.$$

Proof: We consider the following penalized BSDE:

$$\tilde{Y}_t^m = \xi + \int_t^T \hat{f}(s)s + m\int_t^T (L_s - \tilde{Y}_s^m)^+ s - \int_t^T \tilde{Z}_s^m B_s. \tag{26}$$

Comparing it with the penalized BSDE:

$$\hat{Y}_t^m = \xi + \int_t^T \hat{f}(s)s + m\int_t^T (L_s + \int_0^t f(s)s - \hat{Y}_s^m)^+ s - \int_t^T \tilde{Z}_s^m B_s. \tag{27}$$

we know that we only need to prove $\hat{Y}_t^m \to \hat{Y}$, then we have $\tilde{Y}_t^m \to Y$.

Suppose $\{(\tilde{Y}^m, \tilde{Z}^m)\}_{m=1}^\infty$ converges to $(\tilde{Y}, \tilde{Z})$, then $\tilde{Y}$ is the smallest $\hat{f}$-supersolution that dominates L with terminal condition ξ. Next we prove that $(\tilde{Y}, \tilde{Z}) = (Y, Z)$. Apply Itô's formula to $|Y_t^m - \tilde{Y}_t^m|^2$, we have:

$$\begin{aligned}\mathrm{E}|Y_t^m - \tilde{Y}_t^m|^2 + \mathrm{E}\int_t^T |Z_s^m - \tilde{Z}_s^m|^2 \mathrm{d}s = {} & 2\mathrm{E}\int_t^T (Y_s^m - \tilde{Y}_s^m)(f_m(s, Y_s^m, Z_s^m) - \hat{f}(s))\mathrm{d}s \\ & + 2m\mathrm{E}\int_t^T (Y_s^m - \tilde{Y}_s^m)((L_s - Y_s^m)^+ - (L_s - \tilde{Y}_s^m)^+)\mathrm{d}s.\end{aligned}$$

It's easy to check that $(Y_s^m - \tilde{Y}_s^m)((L_s Y_t^m)^+ - (L_s - \tilde{Y}_s^m)^+) \leq 0$. Then we have:

$$\begin{aligned}\mathrm{E}|Y_t^m - \tilde{Y}_t^m|^2 + \mathrm{E}\int_t^T |Z_s^m - \tilde{Z}_s^m|^2 \mathrm{d}s &\leq 2\mathrm{E}\int_t^T (Y_s^m - \tilde{Y}_s^m)(f_m(s, Y_s^m, Z_s^m) - \hat{f}(s))\mathrm{d}s \\ &\leq 2\mathrm{E}\int_t^T (|Y_s^m - Y_s| + |\tilde{Y}_s^m - \tilde{Y}_s|)|f_m(s, Y_s^m, Z_s^m) - \hat{f}(s)|\mathrm{d}s \\ &+ 2\mathrm{E}\int_t^T (Y_s - \tilde{Y}_s)(f_m(s, Y_s^m, Z_s^m) - \hat{f}(s))\mathrm{d}s.\end{aligned}$$

We know $|Y^m - Y| + |\tilde{Y}^m - \tilde{Y}| \to 0$ in $L^2_{\mathscr{F}}$ and $|f_m(s, Y_s^m, Z_s^m) - \hat{f}(s)|$ is uniformly bounded in $L^2_{\mathscr{F}}$. Moreover from the strong convergence of $\{Y^m\}_{m=1}^\infty$ to Y and weak convergence of $\{Z^m\}_{m=1}^\infty$ to Z in $L^2_{\mathscr{F}}$, we know that $\{f_m(\cdot, Y^m, Z^m)\}_{m=1}^\infty$ converges weakly to $\hat{f}(\cdot)$. Thus the right side of the above inequality converges to zero, it follows that $\tilde{Y} \equiv Y$ and $\tilde{Z} \equiv Z$.

□

Follow the Theorem 4.1 d) ⇒ e) in Peng and Xu [10], we can derive directly that $\hat{Y}$ defined in Proposition 4.6 ii) satisfies the following condition: for each $\hat{L}^* \in S^2_{\mathscr{F}}$ such that $\hat{Y} \leq \hat{L}^* \leq \hat{L}$, a.e. a.s.,

$$\int_0^T (\hat{Y}_{t-} - \hat{L}^*_{t-})A_t = 0 \qquad \text{a.s.} \tag{28}$$

Thus, we get the Skorohod condition of $\hat{Y}$ which is our desired result.

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